

LOWER BOUNDS FOR RESONANCE COUNTING FUNCTIONS FOR OBSTACLE SCATTERING IN EVEN DIMENSIONS

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ABSTRACT. In even dimensional Euclidean scattering, the resonances lie on the logarithmic cover of the complex plane. This paper studies resonances for obstacle scattering in \mathbb{R}^d with Dirichlet or admissible Robin boundary conditions, when d is even. Set $n_m(r)$ to be the number of resonances with norm at most r and argument between $m\pi$ and $(m+1)\pi$. Then $\limsup_{r \rightarrow \infty} \frac{\log n_m(r)}{\log r} = d$ if $m \in \mathbb{Z} \setminus \{0\}$.

1. INTRODUCTION

This paper studies resonances for scattering by obstacles in even dimensions. In this setting the resonances lie on the logarithmic cover of the complex plane. The main result is that for $m \in \mathbb{Z} \setminus \{0\}$ the counting function for the number of resonances with norm at most r and argument between $m\pi$ and $(m+1)\pi$ has maximal order of growth. To the best of our knowledge, the only specific obstacles for which this has been known before are balls.

Let $\mathcal{O} \subset \mathbb{R}^d$ be a bounded open set with smooth boundary $\partial\mathcal{O}$, and suppose $\mathbb{R}^d \setminus \mathcal{O}$ is connected. When \mathcal{O} satisfies these conditions we shall call it an *obstacle*. Consider $-\Delta$ on $\mathbb{R}^d \setminus \overline{\mathcal{O}}$, where $\Delta \leq 0$ is the usual Euclidean Laplacian. We impose either Dirichlet ($u|_{\partial(\mathbb{R}^d \setminus \overline{\mathcal{O}})} = 0$) or Robin-type boundary conditions:

$$h(x)u(x) + \frac{\partial}{\partial n}u(x) = 0 \text{ on } \partial(\mathbb{R}^d \setminus \overline{\mathcal{O}})$$

where n is the outward pointing unit normal to $\partial(\mathbb{R}^d \setminus \overline{\mathcal{O}})$ and $h \in C^1(\partial\mathcal{O})$ satisfies $h \geq 0$ in order to be admissible. We note that by choosing $h \equiv 0$ we obtain the Neumann boundary condition. We shall denote the corresponding operator (satisfying either the Dirichlet or admissible Robin-type boundary condition) by P . We choose the upper half-plane to be the physical half plane, so that $R(\lambda) = (P - \lambda^2)^{-1}$ is bounded on $L^2(\mathbb{R}^d \setminus \overline{\mathcal{O}})$ for $\text{Im } \lambda > 0$. It is well known that for any $\chi \in L^\infty_{\text{comp}}(\mathbb{R}^d \setminus \overline{\mathcal{O}})$, $\chi R(\lambda)\chi$ has a meromorphic continuation to \mathbb{C} if d is odd and to Λ , the logarithmic cover of $\mathbb{C} \setminus \{0\}$, if d is even (e.g. [30]). If $\chi = 1$ in a neighborhood of $\overline{\mathcal{O}}$ the location of the poles of $\chi R(\lambda)\chi$ is independent of the choice of such χ . The poles of this meromorphic continuation are called *resonances*.

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We can describe a point $\lambda \in \Lambda$ by its modulus $|\lambda|$ and its argument $\arg \lambda$. On Λ we do not identify points whose arguments differ by an integral multiple of 2π . We define, for $m \in \mathbb{Z}$,

$$\Lambda_m = \{\lambda \in \Lambda : m\pi < \arg \lambda < (m+1)\pi\}$$

and call this the m th sheet of Λ . We note that our choice of the physical half plane means that it is identified with Λ_0 in the case of even d .

For $m \in \mathbb{Z}$ and even d we define the m th resonance counting function:

$$n_m(r) = \#\{\lambda_j : \lambda_j \text{ is a pole of } R(\lambda) \text{ with } m\pi < \arg \lambda_j < (m+1)\pi, \text{ and } |\lambda_j| < r\}.$$

Here and everywhere we count with multiplicity, see (1.1).

Then our main result is

Theorem 1.1. *Let d be even, $\mathcal{O} \subset \mathbb{R}^d$ be an obstacle as defined above, and suppose $\mathcal{O} \neq \emptyset$. Consider $-\Delta$ on $\mathbb{R}^d \setminus \overline{\mathcal{O}}$ with any of Dirichlet, Neumann, or admissible Robin-type boundary conditions. Then, with $n_m(r)$ the resonance-counting function for the m th sheet as defined above,*

$$\limsup_{r \rightarrow \infty} \frac{\log n_m(r)}{\log r} = d$$

for any $m \in \mathbb{Z} \setminus \{0\}$.

The quantity $\limsup_{r \rightarrow \infty} \frac{\log n_m(r)}{\log r}$ is called the order of n_m . By results of Vodev [34, 35], d is the maximum possible value of the order of n_m . A result of Sjöstrand and Zworski [31] for asymptotics of resonances in certain regions of $\Lambda_{\pm 1}$ for the Dirichlet problem for convex obstacles has as a corollary that the order of $n_{\pm 1}(r)$ is at least $d - 1$ for such cases.

In general lower bounds on resonance counting functions have proved elusive. We may contrast Theorem 1.1 with what is known for obstacle scattering in odd dimensions d , $d \geq 3$. In odd dimensions the resonances lie on \mathbb{C} as the double cover of \mathbb{C} . The analogous counting function $n(r)$ is for the number of resonances with norm at most r , and there is an upper bound of the form Cr^d for sufficiently large r , with constant C depending on the obstacle [21]. In *odd* dimensions as far as we know the only specific obstacles for which it is known that $\limsup_{r \rightarrow \infty} (\log n(r)/\log r) = d$ are balls, for which stronger results are known [32, 39]. However, from [6] it is known that this limit must be d for many star-shaped obstacles.

This paper also contains, for completeness, some basic results on the behavior of the scattering matrix $S(\lambda)$ at 0 in even dimensions. In particular, for a large class of operators (“black box” compactly supported perturbations of the Laplacian) in *even* dimension d it is shown that $\lim_{\lambda \downarrow 0} S(\lambda) = I$. See Section 6 for results, references, and further remarks.

Let P be the Laplacian in the exterior of an obstacle with Dirichlet or admissible Robin boundary conditions. Let $S(\lambda)$ denote the scattering matrix associated with the operator P ; its definition is recalled in Section 3. In both even and odd dimensions, when $\arg \lambda = \pi/2$ it is easier to say something about the eigenvalues of $S(\lambda) - I$ than it is for an arbitrary value of λ with $0 < \arg \lambda < \pi$. Lax and Phillips [17], Beale [1], and Vasy [33] used this for *odd* $d \geq 3$ to obtain lower bounds of the type $c_0 r^{d-1}$ for the number of pure imaginary resonances with norm at most r for obstacle scattering ([1, 17]) and for scattering by fixed-sign potentials ([17, 33]). The situation is quite different for *even* dimensions d . For obstacle scattering of the type we consider here or for scattering by fixed-sign potentials for any $m \in \mathbb{Z}$ there are at most finitely many resonances with argument $\pi m + \pi/2$ [1, 9].

Here we make use of the behavior of $S(e^{i\pi/2}\sigma) - I$, $\sigma > 0$, to prove our theorem.

We now outline the proof of our main theorem, which requires introducing some notation and results from other works. With $R(\lambda)$ denoting the meromorphic continuation of the resolvent $(P - \lambda^2)^{-1}$, for $\lambda_0 \in \Lambda$, the multiplicity of a pole of R at λ_0 is defined to be

$$(1.1) \quad \mu_R(\lambda_0) \stackrel{\text{def}}{=} \text{rank} \int_{\gamma_{\lambda_0}} R(\lambda) d\lambda$$

where γ_{λ_0} is a small positively oriented curve enclosing λ_0 and no poles of the resolvent, except, possibly, at λ_0 . For a scalar meromorphic function f defined on Λ , $\lambda_0 \in \Lambda$, we define $m_{\text{sc}}(\lambda_0) = k \in \mathbb{Z}$ if and only if $f(\lambda)(\lambda - \lambda_0)^{-k}$ is bounded in a sufficiently small neighborhood of λ_0 and $\lim_{\lambda \rightarrow \lambda_0} (f(\lambda)(\lambda - \lambda_0)^{-k}) \neq 0$. Thus m_{sc} is positive at zeros and negative at poles, and $m_{\text{sc}}(f, \lambda_0) = 0$ if λ_0 is neither a zero nor a pole of f .

Proposition 1.2. [9, Corollary 4.9] *For P as above, $m \in \mathbb{N}$, and $\lambda_1 \in \Lambda$,*

$$\mu_R(\lambda_1 e^{im\pi}) - \mu_R(\lambda_1) = m_{\text{sc}}(\det(mS(\lambda) - (m-1)I), \lambda_1).$$

Analogues of this result are well known in odd dimensions and for even dimensions for $m = 1$ and a limited subset of Λ ; see, for example, [14, 25, 27, 29].

The restrictions we have put on the operator P mean that neither $R(\lambda)$ nor the scattering matrix has any poles in the physical region Λ_0 . Thus we see from Proposition 1.2 that to study the resonances on Λ_m , $m \in \mathbb{N}$, it will suffice to study the zeros of the scalar function

$$(1.2) \quad f_m(\lambda) \stackrel{\text{def}}{=} \det(mS(\lambda) - (m-1)I)$$

on the physical sheet Λ_0 , which we shall identify with the upper half plane of \mathbb{C} . The function f_m is holomorphic in this region.

In the next two sections we establish some properties of $f_m(\lambda)$ in the upper half plane (corresponding to Λ_0), and on its boundary. In Section 2 we show that if $\arg \lambda = 0$ or $\arg \lambda = \pi$, then $|f_m(\lambda)| = O(|\lambda|^{d-1})$ when $|\lambda| \rightarrow \infty$. In Section 3 we prove that there are constants $M, c_0 > 0$ so that for $\sigma > M > 0$, $\log |f_m(e^{i\pi/2}\sigma)| \geq c_0 \sigma^d$.

Section 4 recalls a result of Govorov [12] for functions analytic in a half-plane and proves a consequence of this result which we shall need in the proof of the theorem. Section 5 proves Theorem 1.1 by showing that the properties of f_m established in Sections 2 and 3 are inconsistent with having $\limsup_{r \rightarrow 0} ((\log r)^{-1} \log n_m(r)) < d$.

Although the proof is different, both the result and some of the ideas underlying the proof of Theorem 1.1 are similar to the results of [7]. The paper [7] shows that for scattering by fixed-sign potentials in even dimensions a lower bound like that of Theorem 1.1 holds. In both [7] and this paper, we study resonances on Λ_m by studying zeros of a function analytic on Λ_0 . We use different complex-analytic results in the two papers – compare Govorov’s results [12, Theorem 3.3] recalled here in Theorem 4.1, to the results of [7, Proposition 2.4]. Additionally, the results we need for the behavior of $f_m(\lambda) = \det(mS(\lambda) - (m-1)I)$ on the boundary of Λ_0 , proved here in Section 2, are much more delicate in the obstacle case than the corresponding results used in [7].

We note that the paper [8] proved, for a Schrödinger operator with a “generic” potential $V \in L_0^\infty(\mathbb{R}^d)$, lower bounds on the m th resonance counting function of the type we prove here.

Section 6 gives, for completeness, some basic results about the behavior of $S(\lambda)$ near $|\lambda| = 0$.

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2. A BOUND ON $|\det(mS(\lambda) - (m-1)I)|$ FOR $\arg \lambda = 0$ OR $\arg \lambda = \pi$

This section uses some results of [10, 11, 18] on the one-sided accumulation of the eigenvalues of the scattering matrix $S(\lambda)$ when $\arg \lambda = 0$ and a sort of “inside-outside duality”. Here “inside-outside duality” refers to a relation between the spectrum of the scattering matrix for the exterior problem (on $\mathbb{R}^d \setminus \overline{\mathcal{O}}$) and the spectrum of the interior operator, that is, the Laplacian with corresponding boundary conditions on \mathcal{O} . To make then summary of the results which we shall need more readable, we present them as two separate theorems, one for the Dirichlet boundary condition and one for the Robin boundary condition.

In this section we identify Λ_0 with the open upper half plane, and similarly identify boundary points. Hence $\lambda \in \mathbb{R}_+$ corresponds to a point in Λ with argument 0. Recall that $S(\lambda)$ is a unitary operator for $\lambda > 0$.

Theorem 2.1. [10, 11, 18] *Let $S(\lambda)$ denote the scattering matrix for $-\Delta$ on $\mathbb{R}^d \setminus \overline{\mathcal{O}}$ with Dirichlet boundary conditions. Let $\lambda, \lambda_0 \in \mathbb{R}_+, \epsilon > 0$. Then $S(\lambda)$ has only finitely many eigenvalues with positive imaginary part. Moreover, $S(\lambda)$ has an eigenvalue $E(\lambda)$ depending continuously on $\lambda \in (\lambda_0 - \epsilon, \lambda_0)$ with $\lim_{\lambda \uparrow \lambda_0} E(\lambda) = 1$*

and $\operatorname{Im} E(\lambda) > 0$ for all $\lambda \in (\lambda_0 - \epsilon, \lambda_0)$ if and only if λ_0^2 is an eigenvalue of $-\Delta$ on \mathcal{O} with Dirichlet boundary conditions. Moreover, there is no pair $\lambda_0, E(\lambda)$ of $\lambda_0 > 0$ and eigenvalue $E(\lambda)$ of $S(\lambda)$ depending continuously on $\lambda \in (\lambda_0, \lambda_0 + \epsilon)$ so that $\lim_{\lambda \downarrow \lambda_0} E(\lambda) = 1$ and $\operatorname{Im} E(\lambda) > 0$ for all $\lambda \in (\lambda_0, \lambda_0 + \epsilon)$.

The results for the Robin-type boundary condition are similar, but the direction of accumulation of the eigenvalues and of the limits is different. In the statement of the theorem, one should understand that if the boundary condition for the exterior problem is

$$h(x)u(x) + \frac{\partial}{\partial n}u(x) = 0 \text{ on } \partial(\mathbb{R}^d \setminus \overline{\mathcal{O}})$$

with n the outward pointing unit normal to $\partial(\mathbb{R}^d \setminus \overline{\mathcal{O}})$, then the boundary condition for the interior problem is

$$h(x)u(x) + \frac{\partial}{\partial n}v(x) = 0 \text{ on } \partial(\overline{\mathcal{O}})$$

where n remains the outward pointing unit normal to $\partial(\mathbb{R}^d \setminus \overline{\mathcal{O}})$.

Theorem 2.2. [11, 18] *Let $S(\lambda)$ denote the scattering matrix for $-\Delta$ on $\mathbb{R}^d \setminus \overline{\mathcal{O}}$ with Robin-type boundary conditions for an admissible function $h \in C^1(\partial\mathcal{O})$, $h \geq 0$. Let $\lambda, \lambda_0 \in \mathbb{R}_+$. Then $S(\lambda)$ has only finitely many eigenvalues with negative imaginary part. Moreover, $S(\lambda)$ has an eigenvalue $E(\lambda)$ depending continuously on $\lambda \in (\lambda_0, \lambda_0 + \epsilon)$ with $\lim_{\lambda \downarrow \lambda_0} E(\lambda) = 1$ and $\operatorname{Im} E(\lambda) < 0$ for all $\lambda \in (\lambda_0, \lambda_0 + \epsilon)$ if and only if λ_0^2 is an eigenvalue of $-\Delta$ on \mathcal{O} with Robin-type boundary conditions. Moreover, there is no pair $\lambda_0, E(\lambda)$ of $\lambda_0 > 0$ and eigenvalue $E(\lambda)$ of $S(\lambda)$ depending continuously on $\lambda \in (\lambda_0 - \epsilon, \lambda_0)$ so that $\lim_{\lambda \uparrow \lambda_0} E(\lambda) = 1$ and $\operatorname{Im} E(\lambda) < 0$ for $\lambda \in (\lambda_0 - \epsilon, \lambda_0)$.*

Results on the one-sided accumulation of eigenvalues of the scattering matrix can be found in [10, 11, 18], with related results in, for example, [36]. The “interior-exterior duality” part of Theorem 2.1 was proved in dimension 2 in [10], and then Theorem 2.2 was proved for the Neumann boundary condition, again in dimension $d = 2$, in [11]. The paper [18] proves both Theorems 2.1 and 2.2 in dimension $d = 3$. However, the proof of [18] works in general dimension $d \geq 2$ with straightforward modifications.

This next proposition is central to the proof of Theorem 1.1.

Proposition 2.3. *Let S be the scattering matrix for the operator P (with either the Dirichlet or admissible Robin boundary condition in the exterior of an obstacle \mathcal{O}) and let $r > 0$. Let $\{e^{i\theta_\alpha(r)}\}_{\alpha \in \mathcal{A}} = \{e^{i\theta_\alpha(r)}\}_{\alpha \in \mathcal{A}(r)}$ be the eigenvalues of $S(r)$, repeated according to their multiplicity. Then*

$$\sum_{\alpha \in \mathcal{A}} \left(\inf_{k \in \mathbb{Z}} \{|\theta_\alpha(r) - 2\pi k|\} \right) = O(r^{d-1}) \text{ as } r \rightarrow \infty.$$

Before proving the proposition, we make a comment about the choice of notation $\{e^{i\theta_\alpha(r)}\}_{\alpha \in \mathcal{A}}$. The eigenvalues of $S(r)$ are a countable set. However, in our proof it will be convenient to choose the phases $\theta_\alpha(r)$ to be continuous functions of r when possible. This is possible when $e^{i\theta_\alpha(r)}$ is away from 1. But it can happen that there is an $r_0 > 0$ and an eigenvalue $E(r)$ of $S(r)$, chosen continuous on $(r_0 - \epsilon, r_0)$, so that $\lim_{r \uparrow r_0} E(r) = 1$ but 1 is *not* an eigenvalue of $S(r_0)$. See [10, Section 2] for examples and further discussion. Hence an eigenvalue of the scattering matrix can “disappear.” With the notation $\{e^{i\theta_\alpha(r)}\}_{\alpha \in \mathcal{A}(r)}$ we wish to indicate the possibility of using different indexing sets for different values of r .

Proof. There is a great deal of flexibility in choosing the set of “phases” $\{\theta_\alpha(\lambda)\}$; note that making a different choice (for example, adding an integral multiple of 2π to one or more of the phases) does not change the value of the sum in the statement of the proposition. To prove the proposition, we shall make a convenient choice of this set.

For $\lambda > 0$, let $\{e^{i\theta_\alpha(\lambda)}\}$ be the eigenvalues of $S(\lambda)$. It is possible to make a choice of the set $\{\theta_\alpha(\lambda)\}$ so that each θ_α is a continuous function of λ , except, perhaps, where $e^{i\theta_\alpha(\lambda)}$ is 1 or approaches 1. In the proof we choose each “phase” $\theta_\alpha(\lambda)$ be continuous as a function of λ except, possibly, at points where a one-sided limit of $e^{i\theta_\alpha(\lambda)}$ is 1. Moreover, we choose each phase to be defined on a maximal open interval in $(0, \infty)$, so that if θ_α is defined on (λ_0, λ_1) , $0 \leq \lambda_0 < \lambda_1 \leq \infty$, if $\lambda_0 > 0$ then $\lim_{\lambda \downarrow \lambda_0} e^{i\theta_\alpha(\lambda)} = 1$. Similarly, if $\lambda_1 < \infty$, then $\lim_{\lambda \uparrow \lambda_1} e^{i\theta_\alpha(\lambda)} = 1$.

Additionally, we require that $\theta_\alpha(\lambda) \in (-2\pi, 2\pi)$. We may impose another condition on the set $\{\theta_\alpha\}$. We require that if $\theta_{\alpha_0}(\lambda)$ is continuous on (λ_0, λ_1) , $0 \leq \lambda_0 < \lambda_1$, and $e^{i\theta_{\alpha_0}(\lambda)}$ is not 1 on that same interval, but $\lim_{\lambda \downarrow \lambda_0} e^{i\theta_{\alpha_0}(\lambda)} = 1$, then $\lim_{\lambda \downarrow \lambda_0} \theta_{\alpha_0}(\lambda) = 0$. We note that in particular this means that if θ_α is defined on $(0, \lambda_0)$, some $\lambda_0 > 0$, then $\lim_{\lambda \downarrow 0} \theta_\alpha(\lambda) = 0$; see Corollary 6.3.

Having chosen these conventions,

$$(2.1) \quad \frac{1}{2\pi i} \int_0^r \frac{d}{d\lambda} \log \det S(\lambda) d\lambda = \frac{1}{2\pi} \sum \theta_\alpha(r) + N(2\pi, r) - N(-2\pi, r) + O(1)$$

as $r \rightarrow \infty$. Here we use the notation

$$\begin{aligned} N(\pm 2\pi, r) \\ = \#\{\lambda_0, 0 < \lambda_0 \leq r : \lim_{\lambda \uparrow \lambda_0} \theta_\alpha(\lambda) = \pm 2\pi \text{ for some } \alpha, \text{ counted with multiplicity } \}. \end{aligned}$$

Now we specialize to the case of admissible Robin-type boundary conditions. By results of [11, 18] recalled here in Theorem 2.2,

$$\begin{aligned} N(-2\pi, r) + \#\{\alpha \in \mathcal{A}(r) : \theta_\alpha(r) < 0\} \\ = \#\{\lambda : 0 < \lambda \leq r \text{ and } \lambda^2 \text{ is a Robin eigenvalue of } -\Delta \text{ on } \mathcal{O}\} + O(1). \end{aligned}$$

By the well-known Weyl formula for the Laplacian on a bounded open set with smooth boundary, this means

$$(2.2) \quad N(-2\pi, r) + \#\{\alpha \in \mathcal{A}(r) : \theta_\alpha(r) < 0\} = c_d \text{vol}(\mathcal{O}) r^d + O(r^{d-1})$$

where c_d is the d -dimensional Weyl constant. Then

$$\frac{1}{2\pi i} \int_0^r \frac{d}{d\lambda} \log \det S(\lambda) d\lambda \geq \frac{1}{2\pi} \sum_{\theta_\alpha(r) < 0} \theta_\alpha(r) - N(-2\pi, r) + O(1) \geq -c_d \text{vol}(\mathcal{O}) r^d + O(r^{d-1}).$$

On the other hand [4, 5, 20, 22, 28]

$$(2.3) \quad \frac{1}{2\pi i} \int_0^r \frac{d}{d\lambda} \log \det S(\lambda) d\lambda = -c_d \text{vol}(\mathcal{O}) r^d + O(r^{d-1}) \text{ as } r \rightarrow \infty.$$

Thus we must have

$$(2.4) \quad \frac{1}{2\pi} \sum_{\theta_\alpha(r) < 0} \theta_\alpha(r) - N(-2\pi, r) = -c_d \text{vol}(\mathcal{O}) r^d + O(r^{d-1})$$

and this, together with (2.2), means that

$$\sum_{\theta_\alpha(r) < 0} (1 + \frac{1}{2\pi} \theta_\alpha(r)) = O(r^{d-1})$$

or

$$(2.5) \quad \sum_{\theta_\alpha(r) < 0} |\theta_\alpha(r) + 2\pi| = O(r^{d-1}).$$

Using (2.1) and (2.3- 2.5), we obtain that

$$\frac{1}{2\pi} \sum_{\theta_\alpha(r) > 0} \theta_\alpha(r) + N(2\pi, r) = O(r^{d-1}).$$

This finishes the proof of the proposition for the Robin case.

The proof for the Dirichlet case is similar, using Theorem 2.1. \square

Proposition 2.4. *Let $m \in \mathbb{N}$, and $r > 0$. Then there is a constant $C > 0$ so that*

$$1 \leq |\det(mS(r) - (m-1)I)| \leq \exp(Cr^{d-1}) \text{ for } r \text{ sufficiently large}$$

and

$$|\det(mS(re^{i\pi}) - (m-1)I)| \leq \exp(Cr^{d-1}) \text{ for } r \text{ sufficiently large.}$$

We note that the constant C depends on m as well as on \mathcal{O} and the boundary condition.

Proof. We denote the set of eigenvalues of the scattering matrix, repeated according to their multiplicity, by $S(r)$ by $\{e^{i\theta_j(r)}\}_{j \in \mathbb{N}}$. Since this time we do not require continuity properties of θ_j we use the index set \mathbb{N} . Then

$$(2.6) \quad \det(mS(r) - (m-1)I) = \prod (1 + m(e^{i\theta_j(r)} - 1)).$$

Again, we make use of the flexibility in choosing the set $\{\theta_j(r)\}$. Here we do *not* need continuity properties, so we can assume, without loss of generality, that

$$-\pi \leq \theta_j(r) < \pi.$$

We shall split the product in (2.6) into two pieces, depending on the size of $|\theta_j(r)|$. From Proposition 2.3, the number of j so that $|\theta_j(r)| > \epsilon > 0$ is $O_\epsilon(r^{d-1})$. Since $1 \leq |1 + m(e^{i\theta_j} - 1)| \leq 1 + 2m$,

$$0 \leq \log \left| \prod_{|\theta_j(r)| > 1/8m} (1 + m(e^{i\theta_j(r)} - 1)) \right| = \sum_{|\theta_j(r)| > 1/8m} \log |(1 + m(e^{i\theta_j(r)} - 1))| = O(r^{d-1}).$$

Now

$$\begin{aligned} \prod_{|\theta_j(r)| \leq 1/8m} (1 + m(e^{i\theta_j(r)} - 1)) &= \exp \left(\sum_{|\theta_j(r)| \leq 1/8m} \log(1 + m(e^{i\theta_j(r)} - 1)) \right) \\ &= \exp \left(\sum_{|\theta_j(r)| \leq 1/8m} (im\theta_j(r) + O(|\theta_j(r)|^2)) \right) \end{aligned}$$

Of course, $\left| \exp \left(\sum_{|\theta_j(r)| < 1/8m} im\theta_j(r) \right) \right| = 1$. Moreover, since if $|\theta_j(r)| < 1/8m$, then $|\theta_j(r)|^2 < |\theta_j(r)|$, and by Proposition 2.3

$$\sum_{|\theta_j(r)| \leq 1/8m} |\theta_j(r)|^2 \leq \sum_{|\theta_j(r)| \leq 1/8m} |\theta_j(r)| = O(r^{d-1})$$

we have that the term $\sum_{|\theta_j(r)| \leq 1/8m} O(|\theta_j(r)|^2) = O(r^{d-1})$. This finishes the proof of the first statement.

To prove the second inequality in the proposition, note that by [9, Proposition 2.1], $S(re^{i\pi}) = 2I - \mathcal{R}S^*(r)\mathcal{R}$, where $\mathcal{R} : L^2(\mathbb{S}^{d-1}) \rightarrow L^2(\mathbb{S}^{d-1})$ is $(\mathcal{R}f)(\theta) = f(-\theta)$, and S^* is the adjoint of S . Hence

$$\begin{aligned} \det(mS(re^{i\pi}) - (m-1)I) &= \det(m(2I - \mathcal{R}S^*(r)\mathcal{R}) - (m-1)I) \\ &= \det((m+1)I - m\mathcal{R}S^*(r)\mathcal{R}) \\ &= \det((m+1)I - mS^*(r)). \end{aligned}$$

Hence, if as before we denote the eigenvalues of $S(r)$ by $\{e^{i\theta_j(r)}\}$, then

$$\det(mS(re^{i\pi}) - (m-1)I) = \prod (1 - m(e^{-i\theta_j(r)} - 1))$$

and the proof of the second inequality follows essentially the same way as the proof of the first one. \square

3. LOWER BOUNDS ON $|\det(mS(\sigma e^{i\pi/2}) - (m-1)I)|$ WHEN $\sigma \rightarrow \infty$, $\sigma \in (0, \infty)$

The main result of this section is Proposition 3.4, which provides a lower bound on $|\det(mS(e^{i\pi/2}\sigma) - (m-1)I)|$ when $\sigma > 0$, $\sigma \rightarrow \infty$. The proof of this proposition uses three main ideas: the fact that $S(i\sigma) - I$ has purely imaginary eigenvalues when d is even and σ is sufficiently large; a monotonicity-type result of [1, 17]; and explicit calculations in the case of a ball along with properties of Bessel functions.

In this section we work in Λ_0 , which we identify with the open upper half plane of \mathbb{C} . Hence for $\sigma > 0$, $i\sigma$ corresponds to $e^{i\pi/2}\sigma$.

In this section we shall make use of some results of [1, 17]. We note that as our choice of the physical half plane is different from theirs (we choose $0 < \arg \lambda < \pi$ as the physical region, they choose $-\pi < \arg \lambda < 0$) some notation will be a bit different.

We recall some basic definitions related to the scattering matrix. For $\lambda \in \mathbb{C}$ with $0 \leq \arg \lambda \leq \pi$ and $\omega \in \mathbb{S}^{d-1}$, there is a unique solution to the equation

$$(-\Delta - \lambda^2)v = 0 \text{ in } \mathbb{R}^d \setminus \mathcal{O}$$

satisfying either the boundary condition (Dirichlet type)

$$v|_{\partial\mathcal{O}} = e^{-i\lambda x \cdot \omega}|_{\partial\mathcal{O}}$$

or satisfying the Robin-type boundary condition

$$h(x)v(x) + \frac{\partial v(x)}{\partial n} = h(x)e^{-i\lambda x \cdot \omega} + \frac{\partial}{\partial n}e^{-i\lambda x \cdot \omega} \text{ on } \partial\mathcal{O}.$$

Here $h \in C^1(\partial\mathcal{O})$, $h \geq 0$, and n is the outward unit normal to $\mathbb{R}^d \setminus \overline{\mathcal{O}}$. In addition, to guarantee uniqueness, we require that v satisfy a radiation condition at infinity: if $\mathcal{O} \subset B(0; R) = \{x \in \mathbb{R}^d : |x| < R\}$, then

$$\left(\frac{\partial}{\partial|x|}v - i\lambda v \right)|_{\mathbb{R}^d \setminus \overline{B(0; R)}} \in L^2(\mathbb{R}^d \setminus \overline{B(0; R)}).$$

It follows then that for large $|x|$, v has the form

$$v(x; \omega, \lambda) = |x|^{-(d-1)/2} e^{i\lambda|x|} (k(\omega, x/|x|, \lambda) + O(|x|^{-1})).$$

This function k is called the *transmission coefficient*.

Now the scattering matrix $S(\lambda)$ is given by $S(\lambda) = I + K(\lambda)$, where

$$(3.1) \quad [K(\lambda)f](\omega) = - \left(\frac{i\lambda}{2\pi} \right)^{(d-1)/2} \int_{\mathbb{S}^{d-1}} k(\omega, -\theta; \lambda) f(\theta) d\theta.$$

The proof of the following lemma uses separation of variables and explicit computations involving Bessel and Hankel functions. Related calculations have been made in many places, including [1, 17, 7, 33].

Lemma 3.1. *Let $\mathcal{O} = B(0; R)$, and let $S(\lambda)$ denote the scattering matrix for $-\Delta$ on $\mathbb{R}^d \setminus \overline{\mathcal{O}}$ with either Dirichlet or Neumann boundary conditions. Let d be even and $m \in \mathbb{N}$. Then there is a constant $c_0 > 0$, depending on R and m , so that for $\sigma > 0$, $|\det(mS(i\sigma) - (m-1)I)| \geq c_0 \exp(c_0 \sigma^d)$.*

Proof. Let $\{Y_l^\mu\}$, $l = 0, 1, 2, \dots$, $\mu = 1, 2, \dots, \mu(l)$ be a complete orthonormal set of spherical harmonics on \mathbb{S}^{d-1} . Here $\mu(l) = \frac{2l+d-2}{d-2} \binom{l+d-3}{d-3}$ and these eigenfunctions of the Laplacian $\Delta_{\mathbb{S}^{d-1}}$ on \mathbb{S}^{d-1} satisfy

$$-\Delta_{\mathbb{S}^{d-1}} Y_l^\mu = l(l+d-2)Y_l^\mu, \quad l = 0, 1, 2, \dots, \mu = 1, 2, \dots, \mu(l).$$

For the Dirichlet Laplacian on $\mathbb{R}^d \setminus \overline{B}(0; R)$, the transmission coefficient $k_D(\lambda)$ is

$$k_D(\theta, \theta', \lambda) = 2 \left(\frac{2\pi}{\lambda} \right)^{(d-1)/2} \sum_{l=0}^{\infty} \sum_{\mu=1}^{\mu(l)} (-i)^l e^{-i(\nu\pi/2 + \pi/4)} \frac{J_\nu(\lambda R)}{H_\nu^{(1)}(\lambda R)} Y_l^\mu(\theta) \overline{Y}_l^\mu(\theta')$$

where $\nu = \nu(l) = l - 1 + d/2$, and J_ν is the Bessel function of order ν of the first kind, and $H_\nu^{(1)}$ is a Hankel function. We are interested in k_D evaluated at $\lambda = i\sigma$. Using [26, 9.6.3 and 9.6.4]

$$k_D(\theta, \theta', i\sigma) = \pi \left(\frac{2\pi}{\sigma} \right)^{(d-1)/2} \sum_{l=0}^{\infty} \sum_{\mu=1}^{\mu(l)} \frac{I_\nu(\sigma R)}{K_\nu(\sigma R)} Y_l^\mu(\theta) \overline{Y}_l^\mu(\theta').$$

Now we note that since the eigenvalues of $k_D(i\sigma)$ are real, and the spherical harmonics are either even or odd in the reflection $\omega \rightarrow -\omega$, the eigenvalues of $K_D(i\sigma)$ are pure imaginary, and the eigenvalues of $(2\pi/\sigma)^{(d-1)/2} K_D(i\sigma)$ have the same norm as the eigenvalues of $k_D(i\sigma)$. Hence

$$\begin{aligned} |\det(mS(i\sigma) - (m-1)I)| &= \prod_l \left(1 + \left| \pi m \frac{I_\nu(\sigma R)}{K_\nu(\sigma R)} \right|^2 \right)^{\mu(l)/2} \\ &= \exp \left[\sum_{l=0}^{\infty} \frac{\mu(l)}{2} \log \left(1 + \left| \pi m \frac{I_\nu(\sigma R)}{K_\nu(\sigma R)} \right|^2 \right) \right] \\ &\geq \exp \left[\sum_{\sigma R/M \leq l \leq \sigma R} \frac{\mu(l)}{2} \log \left(1 + \left| \pi m \frac{I_\nu(\sigma R)}{K_\nu(\sigma R)} \right|^2 \right) \right] \end{aligned}$$

for $M > 1$. From the uniform asymptotic expansions of [26, 9.7.7, 9.7.8], we have that for $\tau > 0$ in a fixed compact set

$$\frac{I_\nu(\nu\tau)}{K_\nu(\nu\tau)} = \frac{1}{\pi} e^{2\nu\eta} (1 + O(\nu^{-1}))$$

where $\eta = \sqrt{1 + \tau^2} + \log(\tau/(1 + \sqrt{1 + \tau^2}))$. By restricting $\sigma R/M \leq l \leq \sigma R$ for some finite $M > 1$, thus ensuring $\tau = \sigma R/\nu$ lies in a compact set away from 0, we get from

these asymptotics that for sufficiently large σ

$$\begin{aligned} |\det(mS(i\sigma) - (m-1)I)| &\geq \exp \left(\sum_{\sigma R/M \leq l \leq \sigma R} \frac{\mu(l)}{2} l \right) \\ &\geq c_0 \sigma^d. \end{aligned}$$

The last inequality uses that $\mu(l) > c'_0 l^{d-2} > 0$ for sufficiently large l .

Likewise, for the Robin-type boundary conditions in the exterior of the sphere where the boundary function is h_0/R , for a constant $h_0 \geq 0$

$$k_{h_0}(\theta, \theta', i\sigma) = \pi \left(\frac{2\pi}{\sigma} \right)^{(d-1)/2} \sum_{l=0}^{\infty} \sum_{\mu=1}^{\mu(l)} \frac{(h_0 + \frac{d-2}{2})I_\nu(\sigma R) - \sigma R I'_\nu(\sigma R)}{(h_0 + \frac{d-2}{2})K_\nu(\sigma R) + \sigma R K'_\nu(\sigma R)} Y_l^\mu(\theta) \bar{Y}_l^\mu(\theta').$$

Thus, a similar computation as in the Dirichlet case, using [26, 9.7.7-9.9.10], gives the result for the Neumann boundary condition ($h_0 = 0$), or indeed for any Robin-type boundary condition with $h_0 \geq 0$. \square

We recall some results of [1, 17] which we shall use. The first is [1, Theorem 3.7] along with some results of [1, Theorem 3.5], which generalizes [17, Theorem 2.4]. In the statement of the theorem we use $k(i\sigma)$ to denote the operator given by $[k(i\sigma)f](\omega) = \int_{\mathbb{S}^{d-1}} k(\omega, \theta; i\sigma) f(\theta) d\theta$. This operator $k(i\sigma)$ is self-adjoint for the boundary conditions which we consider.

Theorem 3.2. [1, Theorem 3.7; see also Theorem 3.5] *Let \mathcal{O}_1 and \mathcal{O}_2 be obstacles so that $\overline{\mathcal{O}_1} \subset \mathcal{O}_2$. Let h_j be admissible boundary functions on $\partial\mathcal{O}_j$, $j = 1, 2$, and let $k_j(\lambda)$ denote the operators on $L^2(\mathbb{S}^{d-1})$ with Schwartz kernels given by the corresponding transmission coefficients for the Robin boundary conditions. Then there is a $\sigma_0 > 0$ depending on \mathcal{O}_1 , \mathcal{O}_2 , and on h_2 so that $0 > k_1(i\sigma) \geq k_2(i\sigma)$ for $\sigma > \sigma_0$. For obstacles \mathcal{O}_1 , \mathcal{O}_2 , $\overline{\mathcal{O}_1} \subset \mathcal{O}_2$, with Dirichlet boundary conditions, $0 < k_1(i\sigma) \leq k_2(i\sigma)$ for all $\sigma > 0$.*

Let $\mathcal{R} : L^2(\mathbb{S}^{d-1}) \rightarrow L^2(\mathbb{S}^{d-1})$ be defined by $(\mathcal{R}f)(\theta) = f(-\theta)$. The following proposition follows immediately from Theorem 3.2 and [17, Theorem 4.4].

Proposition 3.3. *Let \mathcal{O}_j , h_j , and k_j be as in the statement of Theorem 3.2. Then the eigenvalues of $(k_j \mathcal{R})(i\sigma)$ are real for sufficiently large $\sigma > 0$. Order the eigenvalues of $(k_j \mathcal{R})(i\sigma)$, taking account of multiplicities:*

$$\nu_1^{(j)}(i\sigma) \geq \nu_2^{(j)}(i\sigma) \geq \dots > 0 > \dots \geq \kappa_2^{(j)}(i\sigma) \geq \kappa_1^{(j)}(i\sigma), \quad j = 1, 2$$

Then there is a $\sigma_0 \geq 0$ so that for $\sigma > \sigma_0$ and for each $n \in \mathbb{N}$,

$$\nu_n^{(1)}(i\sigma) \leq \nu_n^{(2)}(i\sigma) \text{ and } \kappa_n^{(1)}(i\sigma) \geq \kappa_n^{(2)}(i\sigma).$$

We are now ready to prove the main result of this section.

Proposition 3.4. *Let d be even and let $\mathcal{O} \subset \mathbb{R}^d$ be an obstacle with $\mathcal{O} \neq \emptyset$. Let $S(\lambda)$ denote the scattering matrix of $-\Delta$ on $\mathbb{R}^d \setminus \mathcal{O}$ with either Dirichlet or Robin type boundary conditions. In the latter case assume the boundary function h satisfies $h \in C^1(\partial\mathcal{O})$, $h \geq 0$. Then for $m \in \mathbb{N}$ there is a constant $c > 0$, depending on both \mathcal{O} and the boundary condition, so that for sufficiently large $\sigma > 0$, $|\det(mS(i\sigma) - (m-1)I)| \geq \exp(c|\sigma|^d)$.*

Proof. Our conditions on \mathcal{O} ensure that there is some nontrivial closed ball contained in \mathcal{O} . By translating if necessary, we may assume the ball is $\overline{B}(0; R)$ for some $R > 0$.

We shall apply Proposition 3.3 with $\mathcal{O}_1 = B(0; R)$ and $\mathcal{O}_2 = \mathcal{O}$. We use the original boundary condition on $\mathbb{R}^d \setminus \overline{\mathcal{O}_2} = \mathbb{R}^d \setminus \overline{\mathcal{O}}$. If the original boundary condition is Dirichlet, we use the Dirichlet boundary condition on $\mathbb{R}^d \setminus \overline{\mathcal{O}_1}$; if the original boundary condition is Neumann or Robin type, we use the Neumann boundary condition on $\mathbb{R}^d \setminus \overline{\mathcal{O}_1}$. In each case we denote the corresponding scattering matrices by S_j and the transmission coefficient and its corresponding operator on $L^2(\mathbb{S}^{d-1})$ by k_j .

The eigenvalues of $S_j(i\sigma) - I$ are, by (3.1), given by $-i^{d-1} \left(\frac{\sigma}{2\pi}\right)^{(d-1)/2}$ times the eigenvalues of $k_j(i\sigma)\mathcal{R}$. If we denote the eigenvalues of $k_j(i\sigma)\mathcal{R}$ by $\{\nu_n^{(j)}(i\sigma)\} \cup \{\kappa_j^{(j)}(i\sigma)\}$ with the same ordering as in the statement of Proposition 3.3, we have, for $\sigma > 0$ sufficiently large,

$$\begin{aligned} & |\det(mS(i\sigma) - (m-1)I)| \\ &= |\det(mS_2(i\sigma) - (m-1)I)| \\ &= \prod_{n=1}^{\infty} \left| 1 - mi^{d-1} \left(\frac{\sigma}{2\pi}\right)^{(d-1)/2} \nu_n^{(2)}(i\sigma) \right| \prod_{j=1}^{\infty} \left| 1 - mi^{d-1} \left(\frac{\sigma}{2\pi}\right)^{(d-1)/2} \kappa_j^{(2)}(i\sigma) \right| \end{aligned}$$

Now from Proposition 3.3, for $\sigma > 0$ large enough,

$$\begin{aligned} & |\det(mS_2(i\sigma) - (m-1)I)| \\ &\geq \prod_{n=1}^{\infty} \left| 1 - mi^{d-1} \left(\frac{\sigma}{2\pi}\right)^{(d-1)/2} \nu_n^{(1)}(i\sigma) \right| \prod_{j=1}^{\infty} \left| 1 - mi^{d-1} \left(\frac{\sigma}{2\pi}\right)^{(d-1)/2} \kappa_j^{(1)}(i\sigma) \right| \\ &= |\det(mS_1(i\sigma) - (m-1)I)|. \end{aligned}$$

Since $\mathcal{O}_1 = B(0; R)$, Lemma 3.1 finishes the proof. \square

4. COMPLEX-ANALYTIC RESULTS

In this section we denote by U_+ the upper half plane: $U_+ = \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$. Let $f : U_+ \rightarrow \mathbb{C}$ be an analytic function, not identically 0. Assume in addition that f is continuous on $\overline{U_+} \setminus \{0\}$ and bounded in compact sets of U_+ . Then we define (e.g. [12, page 5] or [19, Section 1.14]) the *order* of f in U_+ to be

$$\rho = \limsup_{r \rightarrow \infty} \frac{\log_+ \log_+ \sup_{z \in U_+, |z| \leq r} |f(z)|}{\log r}.$$

We shall be interested in the case where $\rho > 1$ is finite. When $\infty > \rho > 1$ this definition of order is consistent with the definition of order in an angle given by Govorov in [12, Part I, Section 1, page 1]; see [12, Theorem 1.4].

In preparation for the next theorem, we introduce some notation. For $q \in \mathbb{N}$, let

$$E_q(u) = (1 - u) \exp \left(u + \frac{u^2}{2} + \dots + \frac{u^q}{q} \right)$$

denote the canonical Weierstrass factor. Set, for $a \in \mathbb{C}$, $a \neq 0$,

$$D_q(u, a) = \frac{E_q(u/a)}{E_q(u/\bar{a})},$$

the canonical Nevanlinna factor.

We shall use an adaptation of a result of [12] for a function f analytic on U_+ , of finite order $\rho > 1$, which, in addition, has a continuous extension to \overline{U}_+ . We note that this last condition (the continuous extension to \overline{U}_+) is not made in [12, Theorem 3.3], but allows us to simplify the statement of the result— in particular, with this condition, the singular boundary function, denoted by φ in the statement of [12, Theorem 3.3], is identically constant.

Theorem 4.1. ([12, Theorem 3.3, adapted; see also Theorem 3.2] *Let f be analytic in the half plane U_+ and continuous on \overline{U}_+ . Suppose f is of finite order ρ , $\rho \geq 1$, in U_+ . Let $\{z_n\} = \{r_n e^{i\theta_n}\}$, $0 < \theta_n < \pi$ be the set of zeros of f in U_+ , and set $q = [\rho]$. Then there are real constants a_0, a_1, \dots, a_q so that*

$$(4.1) \quad f(z) = \exp \left(i(a_0 + a_1 z + \dots + a_q z^q) + \frac{1}{\pi i} \int_{-1}^1 \frac{\log |f(t)|}{t - z} dt \right) \prod_{|z_n| \leq 1} \frac{z - z_n}{z - \bar{z}_n} \\ \times \prod_{|z_n| > 1} D_q(z, z_n) \times \exp \left(\frac{z^{q+1}}{\pi i} \int_{|t| \geq 1} \frac{\log |f(t)|}{t^{q+1}(t - z)} dt \right).$$

The integrals and products in this expression are absolutely convergent. Moreover,

$$\sum_{r_n \leq 1} r_n \sin \theta_n < \infty, \quad \sum_{r_n > 1} r_n^{-\rho-\epsilon} \sin \theta_n < \infty, \quad \int_{-\infty}^{\infty} \frac{|\log |f(t)||}{1 + |t|^{1+\rho+\epsilon}} dt < \infty$$

for any $\epsilon > 0$.

We remark for those comparing [12, Theorem 3.3] there seems to be a small error— there does not seem to be a reason that the constants a_j cannot be negative. We include the restriction $\rho > 1$ here because it is for such ρ that our definition of order in a half plane coincides with that of [12].

Proposition 4.2. *Let f be a function analytic in U_+ and continuous on \overline{U}_+ , and of order at most $\rho > 1$ in U_+ . Let*

$$\tilde{n}_f(r) = \#\{a_j \in U_+ : f(a_j) = 0, \text{ counted with multiplicity}\}.$$

Suppose $\tilde{n}_f(r) = O(r^{\rho'})$ as $r \rightarrow \infty$ and $\log |f(\pm t)| = O(t^{\rho'})$ as $t \rightarrow \infty$ for some $\rho' < \rho$. Suppose $[\rho] = q$ is even. Let $\sigma > 0$. Then for any $\epsilon > 0$ there is a constant $C = C_\epsilon < \infty$ so that

$$\log |f(i\sigma)| \leq C(1 + \sigma^{\max(\rho' + \epsilon, q-1)}), \quad \sigma > 0.$$

We comment that the restriction that $[\rho] = q$ be even is necessary. For odd q , we may consider as a counterexample the function $\exp(\pm iz^q)$, where the choice of sign is determined by the parity of $(q-1)/2$.

Proof. In this proof C denotes a positive constant which may depend upon ϵ and may change from line to line.

We use the expression for f from Theorem 4.1, along with the notation of that theorem. In particular, $\{z_n\}$ denotes the set of zeros of f in U_+ , repeated according to their multiplicity. From Theorem 4.1, we can write

$$f(z) = \exp(g_1(z) + g_2(z)) \prod_{|z_n| \leq 1} \frac{z - z_n}{z - \bar{z}_n} \prod_{|z_n| > 1} D_q(z, z_n)$$

where

$$g_1(z) = i(a_0 + a_1 z + \dots + a_q z^q) + \frac{1}{\pi i} \int_{-1}^1 \frac{\log |f(t)|}{t - z} dt$$

and

$$g_2(z) = \frac{z^{q+1}}{\pi i} \int_{|t| \geq 1} \frac{\log |f(t)|}{t^{q+1}(t - z)} dt.$$

Recalling that $a_j \in \mathbb{R}$ and q is even, we see

$$\operatorname{Re} g_1(i\sigma) = O(\sigma^{q-1}) \text{ as } \sigma \rightarrow \infty$$

so that $|\exp(g_1(i\sigma))| \leq C \exp(C\sigma^{q-1})$. Moreover,

$$(4.2) \quad \operatorname{Re}(g_2(i\sigma)) = \frac{i^q \sigma^{q+1}}{2\pi} \int_{|t| \geq 1} \frac{t \log |f(t)|}{t^{q+1}(t^2 + \sigma^2)} dt.$$

Thus, if $\rho' < q-1$, we see immediately that $\operatorname{Re} g_2(i\sigma) = O(\sigma^{q-1})$ since in this case $t^{-q} \log |f(t)|$ is integrable on $\{t \in \mathbb{R} : |t| \geq 1\}$. On the other hand, if $\rho' \geq q-1$, then for $\epsilon > 0$ sufficiently small we can write

$$\begin{aligned} |\operatorname{Re}(g_2(i\sigma))| &\leq \frac{\sigma^{q+1}}{2\pi} \int_{|t| \geq 1} \frac{|\log |f(t)||}{t^{q+1+\rho'-q+\epsilon} \sigma^{1-(\rho'-q+\epsilon)}} dt \\ &\leq C \sigma^{\rho'+\epsilon}. \end{aligned}$$

Consider

$$\prod_{|z_n| > 1} D_q(i\sigma, z_n).$$

We divide this into two cases, depending on the relative size of ρ' and q . We note that if $\rho' < q$, then

$$\sum_{|z_n|>1} \left(\frac{1}{z_n^q} - \frac{1}{\overline{z_n}^q} \right) = \sum_{|z_n|>1} \left(\frac{-2i \operatorname{Im} z_n^q}{|z_n|^{2q}} \right)$$

and that the assumption that $q > \rho'$ implies that the sum converges. Hence, if $\rho' < q$, we have

$$\prod_{|z_n|>1} D_q(i\sigma, z_n) = \left(\prod_{|z_n|>1} D_{q-1}(i\sigma, z_n) \right) \exp \left(-i^{q+1} \sigma^q \sum_{|z_n|>1} \left(\frac{2 \operatorname{Im} z_n^q}{|z_n|^{2q}} \right) \right).$$

Since q is even, the exponent in the second factor is pure imaginary. Now when $\rho' < q$, the estimates used in the proof of [12, Lemma 3.4] (and related to fairly standard estimates of canonical products; compare [19, Section I.4], for example) show that for $\rho' < q$,

$$\left| \prod_{|z_n|>1} D_{q-1}(i\sigma, z_n) \right| \leq C \exp(C|\sigma|^{\max(q-1, \rho'+\epsilon)}), \quad \epsilon > 0.$$

On the other hand, if $\rho' \geq q$, a direct application of the estimates as in the proof of [12, Lemma 3.4] shows that

$$\left| \prod_{|z_n|>1} D_q(i\sigma, z_n) \right| \leq C \exp(C|\sigma|^{\rho'+\epsilon}), \quad \epsilon > 0.$$

In either case, we have

$$\left| \prod_{|z_n|>1} D_q(i\sigma, z_n) \right| \leq C \exp(C|\sigma|^{\max(\rho'+\epsilon, q-1)}), \quad \epsilon > 0.$$

□

5. PROOF OF THEOREM 1.1

In this section we give the proof of Theorem 1.1,. Suppose $m \in \mathbb{N}$. We will apply Proposition 4.2 to $f_m(\lambda) = \det(mS(\lambda) - (m-1)I)$.

We first show that f_m has the regularity properties of Proposition 4.2. Our assumptions on the boundary conditions on $\partial(\mathbb{R}^d \setminus \overline{\mathcal{O}})$ ensure that the resolvent $R(\lambda) = (P - \lambda^2)^{-1}$ is holomorphic in the closure of $\Lambda_0 \subset \Lambda$. Thus $S(\lambda)$ is holomorphic in that region as well. Moreover, $S(\lambda)$ is continuous at $0 \in \{z \in \mathbb{C} : \operatorname{Im} z \geq 0\}$, see Proposition 6.2. Then f_m inherits these regularity properties of the scattering matrix.

The proof of the theorem is by contradiction. Suppose there is some combination of nontrivial obstacle \mathcal{O} and boundary condition (Dirichlet or admissible Robin) so that $\limsup_{r \rightarrow \infty} \frac{\log n_m(r)}{\log r} = \rho' < d$. Let $\tilde{n}_{f_m}(r)$ be the number of zeros of $f_m = \det(mS(\lambda) - (m-1)I)$ in the upper half plane with norm at most r . By

Proposition 1.2, $\limsup_{r \rightarrow \infty} \frac{\log n_m(r)}{\log r} = \limsup_{r \rightarrow \infty} \frac{\log \tilde{n}_{f_m}(r)}{\log r}$. It follows from the same arguments as in, for example, [38, Section 2] or [37, Theorem 7], that the order of $f_m(\lambda) = \det(mS(\lambda) - (m-1)I)$ on the upper half plane is at most d . Then, using Propositions 2.3 and 4.2, we must have $|\det(mS(i\sigma) - (m-1)I)| \leq C \exp(\max(\rho' + \epsilon, d-1))$ for all $\epsilon > 0$ with some constant $C = C_\epsilon$. But this contradicts Proposition 3.4, proving the theorem for $m > 0$.

If $m < 0$, we observe that $n_{-m}(r) = n_m(r)$, as follows, for example, from [9, (2.2)]:

$$S(|\lambda|e^{-i\pi \arg \lambda})^* = 2I - \mathcal{R}S(e^{i\pi} \lambda) \mathcal{R}, \quad \lambda \in \Lambda$$

where $(\mathcal{R}f)(\theta) = f(-\theta)$.

6. THE SCATTERING MATRIX AT 0 IN EVEN DIMENSIONS

The results of this section, while used in the proof of Theorem 1.1, use rather different techniques than the majority of this paper. Hence we include them here so as to not interrupt the flow. We note that both Proposition 6.2 and Corollary 6.3 may be well known, but we are unaware of a reference in which it is proved in this setting. In Section 5 we used the fact that the scattering matrix has continuous extension to $\{z \in \mathbb{C} : \operatorname{Im} z \geq 0\}$. In this section we prove this. With the assumptions we have made on the operator P , the only real issue is the behavior of the scattering matrix at 0. Note that the nature of the singularity of the “model resolvent” $(-\Delta - \lambda^2)^{-1}$ at $\lambda = 0$ depends on the dimension and that the expression for the scattering matrix (for example, [27, Proposition 2.1], recalled here in Proposition 6.1) has dimensional-dependent powers of λ . Thus one expects the scattering matrix to be “more regular” at 0 in higher dimensions; compare, for example, the papers [2, 13, 15, 16] which include much more detailed results on the behavior of the resolvent and scattering matrix at 0 for the Schrödinger operator. However, as our proofs do not depend on the dimension other than through its parity, we give them here for all even dimensions.

The proofs we include here do not require that S be the scattering matrix for the Laplacian with Dirichlet or (admissible) Robin-type boundary conditions in the exterior of an obstacle. In fact, the proofs work for the scattering matrix for any self-adjoint operator P which is a compactly-supported “black-box” perturbation of the Laplacian satisfying the conditions of Sjöstrand-Zworski, see [30]. We recall these assumptions for the reader’s convenience.

In recalling the assumptions of [30] we use similar notation. By a black box operator we mean an operator P defined on a domain $\mathcal{D} \subset \mathcal{H}$ satisfying the conditions below. Let $R_0 > 0$ be fixed, and let $B(R_0) = \{x \in \mathbb{R}^d : |x| < R_0\}$. Let \mathcal{H} be a complex Hilbert space with orthogonal decomposition

$$\mathcal{H} = \mathcal{H}_{B(R_0)} \oplus L^2(\mathbb{R}^d \setminus B(R_0)).$$

Using the notation of [30], we denote the corresponding orthogonal projections by $u \mapsto u|_{B(R_0)}$ and $u \mapsto u|_{\mathbb{R}^d \setminus B(R_0)}$. We assume that the operator $P : \mathcal{H} \rightarrow \mathcal{H}$ is semibounded below and is self-adjoint with domain $\mathcal{D} \subset \mathcal{H}$. Furthermore, if $u \in H^2(\mathbb{R}^d \setminus B(R_0))$ and u vanishes near $B(R_0)$, then $u \in \mathcal{D}$; and conversely $\mathcal{D}|_{\mathbb{R}^d \setminus B(R_0)} \subset H^2(\mathbb{R}^d \setminus B(R_0))$. The operator P is $-\Delta$ outside $B(R_0)$:

$$Pu|_{\mathbb{R}^d \setminus B(R_0)} = -\Delta u|_{\mathbb{R}^d \setminus B(R_0)} \text{ for all } u \in \mathcal{D}$$

and

$$\mathbf{1}_{B(R_0)}(P + i)^{-1} \text{ is compact}$$

where $\mathbf{1}_{B(R_0)}$ is the characteristic function of $B(R_0)$.

We note that the Laplacian on $\mathbb{R}^d \setminus \overline{\mathcal{O}}$ with the boundary conditions we have considered in the main part of the paper satisfy the black box conditions, with \mathcal{H} identified with $L^2(\mathbb{R}^d \setminus \overline{\mathcal{O}})$.

The proof of Proposition 6.2 will use [27, Proposition 2.1] which we recall here for the convenience of the reader. We have adapted the notation somewhat.

Proposition 6.1. ([27, Proposition 2.1]) *For $\phi \in C_c^\infty(\mathbb{R}^d)$, let us denote by*

$$\mathbb{E}_\pm^\phi(\lambda) : L^2(\mathbb{R}^d) \rightarrow L^2(\mathbb{S}^{d-1})$$

the operator with the kernel $\phi(x) \exp(\pm i\lambda \langle x, \omega \rangle)$. Let us choose $\chi_i \in C_c^\infty(\mathbb{R}^d)$, $i = 1, 2, 3$, such that $\chi_i \equiv 1$ near U and $\chi_{i+1} \equiv 1$ on $\text{supp } \chi_i$.

Then for $0 < \arg \lambda < \pi$ we have $S(\lambda) = I + A(\lambda)$, where

$$A(\lambda) = i\pi(2\pi)^{-d} \lambda^{(d-1)/2} \mathbb{E}_+^{\chi_3}(\lambda) [\Delta, \chi_1] R(\lambda) [\Delta, \chi_2]^t \mathbb{E}_-^{\chi_3}(\lambda)$$

where ${}^t\mathbb{E}$ denotes the transpose of \mathbb{E} . The identity holds for $\lambda \in \Lambda$ by analytic continuation.

Proposition 6.2. *Let the dimension d be even, let P be a black box compactly supported perturbation of the Laplacian, and let $S(\lambda)$ be the corresponding scattering matrix, unitary on the positive real axis. Then there is an $\epsilon > 0$ so that $S(\lambda)$ is analytic in $V_\epsilon \stackrel{\text{def}}{=} \{0 \leq \arg \lambda \leq \pi, 0 < |\lambda| < \epsilon\}$, and*

$$\lim_{|\lambda| \rightarrow 0, \lambda \in V_\epsilon} S(\lambda)$$

exists.

Proof. For $0 < \arg \lambda < \pi$ we set $R(\lambda) = (P - \lambda^2)^{-1}$. It is well known (see e.g. [30]) that $\chi \in C_c^\infty(\mathbb{R}^d)$, $\chi R(\lambda) \chi$ has a meromorphic continuation to Λ and that $R(\lambda)$ has only finitely many poles in the region with $0 \leq \arg \lambda \leq \pi$. We are most concerned here with a more delicate analysis near 0, which through Proposition 6.1 will give us information about the scattering matrix near 0.

Let $\chi \in C_c^\infty(\mathbb{R}^d)$. By [23, Theorem 7.9]¹ given $M \in \mathbb{N}$ there are $\beta_j \in \mathbb{N}_0$ and operators $B_{j,k}$, so that near $|\lambda| = 0$, with $0 \leq \arg \lambda \leq \pi$,

$$(6.1) \quad \chi R(\lambda) \chi = \sum_{j=-2}^M \sum_{k=-\beta_j}^{\infty} \lambda^j (\log \lambda)^{-k} B_{j,k} + O(|\lambda|^{M-\delta})$$

if $\delta > 0$. Moreover, the coefficients of the terms which are unbounded at the origin have finite rank. By Proposition 6.1 the scattering matrix has a similar expansion near the origin. In particular, there are at most a finite number of terms which are unbounded near $|\lambda| = 0$, and each has finite rank.

Now we use the fact that for $\arg \lambda = 0$, $|\lambda| > 0$, $S(\lambda)$ is unitary so that $\|S(\lambda)\| = 1$. But this together with the expansion of $S(\lambda)$ near $|\lambda| = 0$ means that the expansion cannot have any terms which are unbounded as $|\lambda| \rightarrow 0$. \square

It was observed in [2] that dimension $d = 2$ for scattering by a Schrödinger operator $-\Delta + V$, $\lim_{\lambda \downarrow 0} S(\lambda) = I$ for any real-valued V satisfying certain decay conditions. This had earlier been noted for Schrödinger operators in dimension $d \geq 4$, see [13, 15]. This contrasts with the case of dimensions $d = 1$ and $d = 3$, e.g. [3, 16]. We show here that a similar phenomena holds in any even dimension for any operator P satisfying the black-box conditions of Sjöstrand-Zworski, including the exterior Laplacians of the type considered in the main body of the paper.

We give below a proof of this which is somewhat algebraic, and hence is rather different from the proof given in [2] for $d = 2$ for Schrödinger operators.²

Corollary 6.3. *Let d be even, let P be any self-adjoint operator satisfying the black-box conditions of Sjöstrand-Zworski recalled above, and let S denote the corresponding scattering matrix. Then $\lim_{\lambda \downarrow 0} S(\lambda) = I$.*

Proof. By Proposition 6.2 we can write $\lim_{\lambda \downarrow 0} S(\lambda) = S(0) = \lim_{\lambda \downarrow 0} S(e^{i\pi} \lambda)$.

For $\lambda > 0$, $S(\lambda)S^*(\lambda) = I$ and

$$(6.2) \quad S^*(\lambda) = 2I - \mathcal{R}S(e^{i\pi} \lambda)\mathcal{R}.$$

By continuity, both of these hold as well with $\lambda = 0$. In particular, $S^*(0)S(0) = I$ and any eigenvalue of $S(0)$ is of the form $e^{i\theta}$ for some $\theta \in \mathbb{R}$. Suppose u is an eigenfunction of $S(0)$ with eigenvalue $e^{i\theta}$, and $\|u\| = 1$. Then

$$(6.3) \quad e^{i\theta} = \langle S(0)u, u \rangle = \langle (2I - \mathcal{R}S^*(0)\mathcal{R})u, u \rangle$$

by (6.2) at $\lambda = 0$. But since $\|\mathcal{R}S^*(0)\mathcal{R}\| \leq 1$, this means

$$|2 - e^{i\theta}| = |\langle \mathcal{R}S^*(0)\mathcal{R}u, u \rangle| \leq \|u\|^2 = 1.$$

¹ See also [2, 13, 15], or [24, Theorem 4.1] which contain more detailed information for more specific cases.

² We note that [2] proved, for Schrödinger operators, a much stronger result than our Corollary 6.3, since [2] finds the first term or two in the asymptotic expansion of $S(\lambda) - I$ at the origin.

Since we can have $|2 - e^{i\theta}| \leq 1$ for $\theta \in \mathbb{R}$ if and only if $e^{i\theta} = 1$, we are done. \square

We note that it is the application of (6.2) in (6.3) that is particular to the even-dimensional case. The analogous relation for the odd-dimensional case is that $S^*(e^{i\pi}\lambda) = \mathcal{RS}(\lambda)\mathcal{R}$ for $\lambda > 0$, leading to the familiar conclusion that if $e^{i\theta}$ is an eigenvalue of $S(0)$ in odd dimensional black-box scattering, then $e^{i\theta} = \pm 1$.

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